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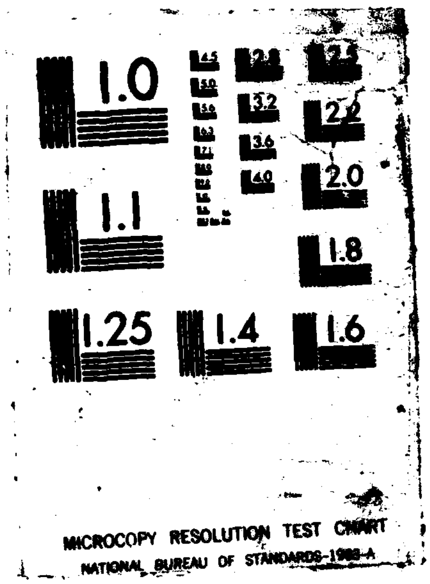
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ON THE ASYMPTOTIC JOINT DISTRIBUTIONS OF THE EIGENVALUES OF RANDOM MATRICES WHICH ARISE UNDER COMPONENTS OF COVARIANCE MODEL

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ABSTRACT

In this paper, the authors derived asymptotic joint distributions of the eigenvalues of some random matrices which arise under components of covariance model.

AMS 1980 subject classifications. Primary 62E20; Secondary 62H15, 62H12.

Key Words and Phrases: Component of covariance model, eigenstructure analysis, limiting distribution, random matrix.

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1. INTRODUCTION

Consider the following components of covariance model

$$\underset{\sim}{X}_{ij} = \underset{\sim}{\mu} + \underset{\sim}{\alpha}_i + \underset{\sim}{\varepsilon}_{ij} \tag{1.1}$$

for $j=1,2,\ldots,m_i$ and $i=1,2,\ldots,k$, where $\underset{\sim}{X}_{ij}$: $p\times 1$, is the j-th observation of the i-th group, $\underset{\sim}{\mu}$ is an unknown general mean vector, $\underset{\sim}{\alpha}_i$ is the random effect vector of the i-th group, and $\underset{\sim}{\varepsilon}_{ij}$ is the error vector. It is assumed that $\underset{\sim}{\alpha}_i$'s are iid., $\underset{\sim}{\varepsilon}_{ij}$'s are iid., $\underset{\sim}{\alpha}_i$'s and $\underset{\sim}{\varepsilon}_{ij}$'s are independent, and that

$$\begin{gathered} E(\underset{\sim}{\alpha}_1) = E(\underset{\sim}{\varepsilon}_{11}) = \underset{\sim}{0}, \\ E(\underset{\sim}{\alpha}_1\underset{\sim}{\alpha}_1') = \Psi, \quad E(\underset{\sim}{\varepsilon}_{11}\underset{\sim}{\varepsilon}_{11}') = \Sigma_1 > 0, \\ E(\underset{\sim}{\alpha}_1'\underset{\sim}{\alpha}_1)^2 + E(\underset{\sim}{\varepsilon}_{11}'\underset{\sim}{\varepsilon}_{11})^2 < \infty, \end{gathered} \tag{1.2}$$

where Ψ is a $p\times p$ non-negative definite matrix. In contrast to the MANOVA model with the non-random effects, which is called the model I, the model (1.1) is also called the model II. If $m_1 = m_2 = \ldots = m_k = m$, the model is balanced, otherwise unbalanced.

Recently, the components of covariance model has received considerable attention in literature. Its importance not only consists in the generality of the model, but also its connection to linear structural

relationship models and the signal processing. For a discussion on applications in linear structural relations, we refer to Anderson (1984) whereas we refer to Zhao, Krishnaiah and Bai (1986b) for applications in signal processing.

Many papers were devoted to the balanced components of covariance models. Anderson, Anderson and Olkin (1986) studied the maximum likelihood estimators (MLE) of $\underset{\sim}{\mu}$, Ψ and Σ_1 under the condition that $\Psi \geq 0$ and is of a maximum rank, and obtained the likelihood ratio criterion (LRC) for testing the rank hypothesis on Ψ. Some of these have been obtained by Theobald (1975), and Schott and Saw (1984). Rao (1983) considered some test procedures for testing relationships between two covariance matrices and these procedures are useful in finding the rank of the component of covariance. Some results for the case of unrestricted covariance matrix were announced by Anderson (1984), where an extensive list of references is given. Some authors put the determination of the rank of Ψ in the framework of model selection; some references of this aspect are given in Zhao, Krishnaiah and Bai (1986 a, 1986 b). In these two papers, they also proposed some procedures to determine the multiplicities of the eigenvalues of the matrix $\Psi\Sigma_1^{-1}$ or Ψ.

For the unbalanced components of covariance model, Bai, Krishnaiah and Zhao (1987) proposed some consistent procedures for the determination of the rank q of the random effect covariance matrix Ψ. One important case under consideration is that Σ_1 is arbitrary definite matrix and $0 \leq q \leq p$. It is assumed that

$$1 \leq m_i \leq M, \qquad i=1,2,\ldots, \tag{1.3}$$

and that

$$k/N \leq \gamma < 1, \tag{1.4}$$

where $N = \sum_{i=1}^{k} m_i$, M and γ are constants, M an integer. We shall keep this assumption in this paper.

Write

$$\underset{\sim}{\overline{X}}_i = \sum_{j=1}^{m_i} \underset{\sim}{X}_{ij}/m_i, \qquad \underset{\sim}{X}_* = \sum_{i=1}^{k} m_i \underset{\sim}{\overline{X}}_i/N, \tag{1.5}$$

and

$$W_1 = \sum_{i=1}^{k} \sum_{j=1}^{m_i} (\underset{\sim}{X}_{ij} - \underset{\sim}{\overline{X}}_i)(\underset{\sim}{X}_{ij} - \underset{\sim}{\overline{X}}_i)',$$

$$W_2 = \sum_{i=1}^{k} m_i (\underset{\sim}{X}_i - \underset{\sim}{X}_*)(\underset{\sim}{X}_i - \underset{\sim}{X}_*)'. \tag{1.6}$$

W_2 and W_1 are called the between groups and within group sums of squares and cross products matrices respectively. It is easily seen that

$$\frac{1}{N-k} E(W_1) = \Sigma_1, \quad \frac{1}{k-1} E(W_2) = \Sigma_1 + \frac{N}{k-1}\left(1 - \frac{1}{N^2} \sum_{i=1}^{k} m_i^2\right)\Psi.$$

We define

$$\Sigma_2 = \Sigma_1 + \frac{N}{k} \Psi, \tag{1.7}$$

which equals $\frac{1}{k-1}E(W_2)$ when $m_1 = m_2 = \ldots = m_k$.

In the above literatures, eigenstructure method occupies an important position. Especially, the study for the eigenstructure of the random matrix $\frac{1}{k-1} W_2 (\frac{1}{N-k} W_1)^{-1}$ is of theoretical and practical interest. In this paper, we shall seek for the limiting joint distribution of its eigenvalues.

For the model I, the joint distributions or asymptotic joint distributions of the multivariate analysis of variance (MANOVA) matrices have been found by some authors. Fisher (1939), Hsu (1939) and Roy (1939) have independently derived the joint distribution of the eigenvalues of the MANOVA matrix in the central case. Hsu (1941) derived the above distribution in the noncentral case when the sample size tends to infinity and the underlying distribution is multivariate normal. Bai, Krishnaiah and Liang (1984) extended this result to the case when the underlying distribution is not necessarily multivariate normal. In proving these results, they all assumed that the ratios of the sample sizes of the groups to the total sample size tend to constants in the limiting case, whereas the number of the groups is fixed. On the contrary, in order to seek for the limiting distribution of the eigenvalues of $\frac{N-k}{k-1}W_2W_1^{-1}$ in the model II, we need to assume that the number k of the groups tends to infinity.

The balanced and unbalanced cases are discussed in Sections 2 and section 3 respectively.

2. THE BALANCED CASE

In this section, we seek for the asymptotic joint distribution of the eigenvalues of the matrix $\frac{N-k}{k-1}W_2W_1^{-1}$ for the balanced case. Here we assume that $k \to \infty$ and

$$m_1 = m_2 = \ldots = m_k = m \geq 2. \tag{2.1}$$

This problem can be reduced to the problem of seeking for the asymptotic joint distribution of the eigenvalues of $S_2S_1^{-1}$, where

$$S_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \underset{\sim}{Y}_i \underset{\sim}{Y}_i', \tag{2.2}$$

$$S_2 = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{Z}_i \underset{\sim}{Z}_i', \tag{2.3}$$

and $\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_{n_1}$, $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ are independent $p\times1$ vectors, $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_{n_1}$ have a common distribution, so do $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$, satisfying the following conditions:

$$\begin{gathered} E\underset{\sim}{Y}_1 = E\underset{\sim}{Z}_1 = \underset{\sim}{0}, \\ E\underset{\sim}{Y}_1\underset{\sim}{Y}_1' = \Sigma_1 > 0, \quad E\underset{\sim}{Z}_1\underset{\sim}{Z}_1' = \Sigma_2 \geq 0, \\ E(\underset{\sim}{Y}_1'\underset{\sim}{Y}_1)^2 < \infty, \quad E(\underset{\sim}{Z}_1'\underset{\sim}{Z}_1)^2 < \infty. \end{gathered} \tag{2.4}$$

Without loss of generality, we can assume that $\Sigma_1 = I_p$ and $\Sigma_2 = \mathrm{diag}$ $[\lambda_1, \ldots, \lambda_p]$ with $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$, where $\lambda_1 \geq \ldots \geq \lambda_p$ are eigenvalues of $\Sigma_2\Sigma_1^{-1}$.

Write

$$\lambda_i = \xi_h, \quad \text{if } i=a_{h-1}+1, \ldots, a_h, \quad h=1,2,\ldots,H,$$

where $\xi_1 > \xi_2 > \ldots > \xi_H \geq 0, \quad a_0=0, \quad a_h = a_{h-1}+\mu_h,$

$$a_H = p = \mu_1 + \ldots + \mu_H.$$

Assume that n_1 varies as n and $\lim_{n\to\infty} n/n_1 = \theta^2$, $0 < \theta^2 < \infty$. Let $\phi_1^{(n)} \geq \ldots \geq \phi_p^{(n)}$ denote the eigenvalues of $S_2S_1^{-1}$. We are interested in the asymptotic joint distribution of $\zeta_i^{(n)}$, $i=1,2,\ldots,p$, where $\zeta_i^{(n)} = \sqrt{n}(\phi_i^{(n)}-\lambda_i)$.

We need the following lemmas:

LEMMA 2.1. Let $\underset{\sim}{X}_n$, $n=0,1,2,\ldots$, be a sequence of random p-vectors with $\underset{\sim}{X}_n \to \underset{\sim}{X}_o$ in distribution. Then there exists a probability space (Ω,F,P) on which we can define a sequence of random vectors $\tilde{\underset{\sim}{X}}_n$, $n=0,1,2,\ldots$, such that

1. $\underset{\sim}{X}_n$ and $\tilde{\underset{\sim}{X}}_n$ are identically distributed.
2. $\tilde{\underset{\sim}{X}}_n \to \tilde{\underset{\sim}{X}}_o$ pointwise.

The above lemma is given in Skorokhod (1956).

LEMMA 2.2. Let $g_n(x)$ be a sequence of K-degree polynomials with roots $x_1^{(n)},\ldots,x_K^{(n)}$ for each n, and let g(x) be a k-degree polynomial with roots $x_1,\ldots,x_k$, $k \leq K$. If $g_n(x) \to g(x)$ as $n \to \infty$, then after suitable rearrangement of $x_1^{(n)},\ldots,x_k^{(n)}$, we have $x_j^{(n)} \to x_j$, $j=1,\ldots,k$ and $|x_j^{(n)}| \to \infty$, $j=k+1,\ldots,K$.

For a proof, we refer to Bai (1984).

Put $n_1 = \theta^{-2}n\beta_n^{-2}$ with $\beta_n \to 1$. The eigenvalues of $S_2S_1^{-1}$ are the roots of the determinant equation

$$\det (S_2-\phi S_1) = 0. \tag{2.5}$$

Write

$$U_n = (u_{ij}^{(n)}) = \sqrt{n_1}(S_1-\Sigma_1) = \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} (\underset{\sim}{Y}_i \underset{\sim}{Y}_i' - I_p),$$

$$V_n = (v_{ij}^{(n)}) = \sqrt{n}(S_2-\Sigma_2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\underset{\sim}{Z}_i \underset{\sim}{Z}_i' - \mathrm{diag}[\lambda_1,\ldots,\lambda_p]).$$

Then (2.5) becomes

$$\det\left(\frac{1}{\sqrt{n}} V_n - \frac{\theta\beta_n}{\sqrt{n}} \phi U_n - D\right) = 0, \tag{2.6}$$

where

$$D = \begin{pmatrix} (\phi-\xi_1)I\mu_1 & & \\ & \ddots & \\ & & (\phi-\xi_H)I\mu_H \end{pmatrix}$$

By the central limit theorem and Lemma 2.1, we can assume that

$$\begin{aligned} U_n &\to U \triangleq (u_{ij}), \quad \text{a.s.} \\ V_n &\to V \triangleq (v_{ij}), \quad \text{a.s.} \end{aligned} \tag{2.7}$$

satisfying the following conditions:

(1) $\{u_{ij}, 1 \le i \le j \le p\}$ and $\{v_{ij}, 1 \le i \le j \le p\}$ are two $p(p+1)/2$ dim.normal vectors independent each other.

(2) $u_{ij} = u_{ji}$, $v_{ij} = v_{ji}$ for all i, j.

(3) $EU_{ij} = Ev_{ij} = 0$ for all i, j.

(4) $\mathrm{cov}(v_{ii},v_{jj}) = EZ_{1i}^2 Z_{ij}^2 - \lambda_i\lambda_j$, if $1 \le i \le j \le p$,

$\mathrm{cov}(u_{ii},u_{jj}) = EY_{1i}^2 Y_{1j}^2 - 1$, if $1 \le i \le j \le p$,

and

$$\mathrm{cov}(v_{i_1i_2},v_{j_1j_2}) = EZ_{1i_1}Z_{1i_2}Z_{1j_1}Z_{1j_2}$$

$$\mathrm{cov}(u_{i_1i_2},u_{j_1j_2}) = EY_{1i_1}Y_{1i_2}Y_{1j_1}Y_{1j_2},$$

if $1 \leq i_1 \leq i_2 \leq p$, $1 \leq j_1 \leq j_2 \leq p$ and at least one of the strict inequalities $i_1 < i_2$ and $j_1 < j_2$ holds. Here Y_{1i} and Z_{1i} are the i-th components of the vectors $\underset{\sim}{Y}_1$ and $\underset{\sim}{Z}_1$, respectively.

Split the matrices U_n, V_n, U and V into blocks as follows

$$U_n = (U_{hg}^{(n)}), \quad V_n = (V_{hg}^{(n)}), \quad U = (U_{hg}), \quad V = (V_{hg}),$$
$$h,g = 1,2,\ldots,H,$$

where $U_{hg}^{(n)}$, $V_{hg}^{(n)}$, U_{hg}, V_{hg} have order $\mu_h \times \mu_g$, $h,g = 1,\ldots,H$.

Take the variable transformation $\phi = \xi_1 + \zeta/\sqrt{n}$. Multiply by $n^{1/4}$ the first μ_1 rows and the first μ_1 columns of the determinant on the left hand side of (2.6), (2.6) can be rewritten as

$$f_n(\zeta) = 0, \tag{2.8}$$

where $f_n(\zeta)$ is a polynomial in ζ with degree p, and

$$\lim_{n\to\infty} f_n(\zeta) = f(\zeta) =$$
$$= \det\begin{pmatrix} V_{11}-\theta\xi_1 U_{11}-\zeta I\mu_1 & & & \\ & (\xi_2-\xi_1)I\mu_2 & & \\ & & \ddots & \\ & & & (\xi_H-\xi_1)I\mu_H \end{pmatrix}. \tag{2.9}$$

After the variable transformation, the p roots of (2.6) (i.e., (2.8)) can be written as $\zeta_{i1}^{(n)} = \sqrt{n}(\phi_i^{(n)}-\xi_1)$. Since $\lim_{n\to\infty} \phi_i^{(n)} = \lambda_i$, $i=1,2,\ldots,p$, we have $\lim_{n\to\infty} \zeta_{i1}^{(n)} = -\infty$, $i=a_1 + 1,\ldots,p$.

By Lemma 2.2,

$$\lim_{n\to\infty} \zeta_{i1}^{(n)} = \zeta_i, \quad i=1,2,\ldots,a_1, \tag{2.10}$$

where $\zeta_1,\ldots,\zeta_{a_1}$ are the roots of

$$\det (V_{11}-\theta\xi_1 U_{11}-\zeta I_{\mu_1}) = 0. \tag{2.11}$$

Similarly, we can prove that the joint distribution of $\zeta_1^{(n)},\ldots,\zeta_p^{(n)}$ tends to that of $\zeta_1,\ldots,\zeta_p$ as $n\to\infty$, where $\zeta_{a_{h-1}+1} \geq \zeta_{a_{h-1}+2} \geq \cdots \geq \zeta_{a_h}$ are the roots of

$$\det (V_{hh}-\theta\xi_h U_{hh}-\zeta I_{\mu_h}) = 0, \quad h=1,2,\ldots,H \tag{2.12}$$

Thus, we have proved the following

THEOREM 2.1. Let S_1,S_2 be defined by (2.2) and (2.3) respectively and let $\phi_1^{(n)},\ldots,\phi_p^{(n)}$ be the eigenvalues of $S_2S_1^{-1}$. Write $\zeta_i^{(n)} = \sqrt{n}(\phi_i^{(n)}-\lambda_i)$, $i=1,2,\ldots,p$. Suppose that $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_{n_1},\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_n$ are independent and $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_{n_1}$ have a common distribution, so do $\underset{\sim}{Z}_1,\ldots\underset{\sim}{Z}_n$, where $E\underset{\sim}{Y}_1 = E\underset{\sim}{Z}_1 = 0$, $E\underset{\sim}{Y}_1\underset{\sim}{Y}_1' = I_p$, $E\underset{\sim}{Z}_1\underset{\sim}{Z}_1' = \operatorname{diag}[\lambda_1,\ldots,\lambda_p]$, $E(\underset{\sim}{Y}_1'\underset{\sim}{Y}_1)^2 < \infty$, $E(\underset{\sim}{Z}_1'\underset{\sim}{Z}_1)^2<\infty$, and that $\lim_{n\to\infty} \frac{n}{n_1} = \theta^2 > 0$. Then the joint distribution of $\zeta_1^{(n)},\ldots,\zeta_p^{(n)}$ tends to that of $\zeta_1,\ldots,\zeta_p$, where $\zeta_{a_{h-1}+1} \geq \cdots \geq \zeta_{a_h}$ are the roots of the determinant equation

$$\det (Q_{hh}-\zeta I_{\mu_h}) = 0, \tag{2.13}$$

and

$$Q_{hh} = (q_{ij}), \quad i,j=a_{h-1}+1,\ldots,a_h, \quad h=1,2,\ldots,H,$$

satisfying

(1) $q_{ij} = q_{ji}$ for each (i,j).

(2) $\{q_{ij},\ a_{h-1}+1 \le i \le j \le a_h,\ h=1,2,\ldots,H\}$ has a joint normal distribution.

(3) $Eq_{ij} = 0$ for each (i,j),

$$\mathrm{Cov}(q_{ii},q_{jj}) = EY_{1i}^2Y_{1j}^2 + \theta^2\xi_h\xi_g EZ_{1i}^2Z_{1j}^2 - \xi_h\xi_g(1+\theta^2),$$

if $\quad a_{h-1} + 1 \le i \le a_h,\quad a_{g-1} + 1 \le j \le a_g,\quad 1 \le h \le g \le H,$

and

$$\mathrm{Cov}(q_{i_1i_2},q_{j_1j_2}) = EY_{1i_1}Y_{1i_2}Y_{1j_1}Y_{1j_2} + \theta^2\xi_h\xi_g EZ_{1i_1}Z_{1i_2}Z_{1j_1}Z_{1j_2},$$

if $a_{h-1}+1 \le i_1 \le i_2 \le a_h,\quad a_{g-1}+1 \le j_1 \le j_2 \le a_g,\quad 1 \le h \le g \le H$ and at least one of the inequalities $i_1 < i_2$, $j_1 < j_2$ holds.

Note that (2.13) is equivalent to (2.12) by the independence between U and V, and Theorem 2.1 holds evidently.

THEOREM 2.2. Besides the assumptions of Theorem 2.1, we assume that

$$EY_{1i}^2Y_{1j}^2 = EY_{1i}^2EY_{1j}^2 = 1,\quad 1 \le i < j \le p,$$

$$EZ_{1i}^2Z_{1j}^2 = EZ_{1i}^2EZ_{1j}^2 = \lambda_i\lambda_j,\quad 1 \le i < j \le p,$$

and

$$EY_{1i_1}Y_{1i_2}Y_{1j_1}Y_{1j_2} = EZ_{1i_1}Z_{1i_2}Z_{1j_1}Z_{1j_2} = 0$$

for all i_1, i_2, j_1, j_2 satisfying that $a_{h-1} + 1 \le i_1 \le i_2 \le a_h$, $a_{g-1} + 1 \le j_1 \le j_2 \le a_h$, $1 \le h \le g \le H$ and that at least one of $i_1 < i_2$, $j_1 < j_2$ holds. Then the limiting vectors $(\zeta_1,\ldots,\zeta_{a1}),\ldots,(\zeta_{a_{H-1}+1},\ldots,\zeta_p)$ are independent each other.

THEOREM 2.3. Under the assumptions of Theorem 2.2 and that

$EY_{1i}^4 = 3$, $EZ_{1i}^4 = 3\lambda_i^2$, $i=1,\ldots,p$, the joint distribution of $\eta_i^{(n)} = \sqrt{n}(\phi_i^{(n)} - \xi_h)/(\sqrt{2(1+\theta^2)}\xi_h)$, $i=a_{h-1}+1,\ldots,a_h$, $h=1,2,\ldots,H$ tends to a limiting distribution with density

$$\prod_{h=1}^{H} D(\eta_{a_{h-1}+1},\ldots,\eta_{a_h}),$$

where $D(x_1,\ldots,x_\mu)$ is given by

$$D(x_1,\ldots,x_\mu) = 2^{-\mu/2}\left(\prod_{i=1}^{\mu} \Gamma\left(\frac{i}{2}\right)\right)^{-1} \left\{\prod_{1\le i<j\le\mu}(x_i - x_j)\right\} \exp\left\{-\frac{1}{2}\sum_{i=1}^{\mu} x_i^2\right\},$$

$$\infty > x_1 \ge x_2 \ge \ldots \ge x_\mu > -\infty.$$

(Refer to Bai, Krishnaiah and Liang, (1984)).

To prove this theorem, it is enough to notice that $[2(1+\theta^2)]^{-1/2}\, Q_{hh}$ is a (real) central Gaussian matrix of order $\mu_h \times \mu_h$ and that $D(x_1,\ldots,x_\mu)$ is the joint density of the eigenvalues $\nu_1 \ge \ldots \ge \nu_\mu$ of a central Gaussian matrix of order $\mu\times\mu$. For the definition of Gaussian matrix, refer to Zhao, Krishnaiah and Bai(1987).

3. THE UNBALANCED CASE

In this section, we study the asymptotic joint distribution of the eigenvalues of $\frac{N-k}{k-1}W_2W_1^{-1}$ defined by (1.6). For simplicity, we assume that $\underset{\sim}{\alpha}_i$ and $\underset{\sim}{\varepsilon}_{ij}$ in (1.1) are distributed as p×1 normal vectors for each (i,j), and that

$$\begin{aligned}&\alpha_i\text{'s are iid., } \varepsilon_{ij}'\text{s are iid., } \alpha_i\text{'s and } \varepsilon_{ij}'\text{s are}\\ &\text{independent, } E(\underset{\sim}{\alpha}_1) = E(\underset{\sim}{\varepsilon}_{11}) = \underset{\sim}{0}, \text{ and}\\ &\quad E(\underset{\sim}{\alpha}_1\underset{\sim}{\alpha}_1') = \Psi \;, \quad E(\underset{\sim}{\varepsilon}_{11}\underset{\sim}{\varepsilon}_{11}') = \Sigma_1 > 0.\end{aligned} \tag{3.1}$$

Besides, we assume that

$$\frac{k}{N-k} = \theta^2 + o(\frac{1}{\sqrt{k}}), \quad \theta > 0, \text{ a constant,} \tag{3.2}$$

$$\frac{1}{k}\sum_{i=1}^{k} m_i^2 = m^2 + O(\frac{1}{\sqrt{k}}), \quad 1 \le m_i \le M, \quad i=1,\dots,k, \tag{3.3}$$

m and M are constants.

Let $\delta_1^{(k)} \ge \dots \ge \delta_p^{(k)}$, $\psi_1 \ge \dots \ge \psi_p$ and $\lambda_1^{(k)} \ge \dots \ge \lambda_p^{(k)}$ denote the eigenvalues of $\frac{1}{k-1}W_2(\frac{1}{N-k}W_1)^{-1}$, $\Psi\Sigma_1^{-1}$ and $\Sigma_2\Sigma_1^{-1}$ respectively, where

$$\Sigma_2 = \Sigma_1 + \frac{N}{k}\Psi = \Sigma_1 + (1+\theta^{-2})\Psi + o(\frac{1}{\sqrt{k}})\Psi.$$

Without loss of generality, we can assume that

$$\Sigma_1 = Ip \text{ and } \Psi = \text{diag}\,[\psi_1,\dots,\psi_p].$$

Write

$$\lambda_i^{(k)} = \xi_h^{(k)}, \quad \psi_i = \bar{\psi}_h, \quad \text{if } i=a_{h-1}+1,\dots,a_h, \quad h=1,2,\dots,H,$$

where $\tilde{\psi}_1 > \tilde{\psi}_2 > \dots > \tilde{\psi}_H \geq 0$, $\xi_h^{(k)} = 1 + \frac{N}{k}\tilde{\psi}_h$, $h=1,2,\dots,H$.

The eigenvalues of $\frac{1}{k-1}W_2(\frac{1}{N-k}W_1)^{-1}$ are the roots of the determinant equation

$$\det\left(\frac{1}{k-1}W_2 - \frac{\delta}{N-k}W_1\right) = 0. \tag{3.4}$$

Put $N-k = \theta^{-2}k\beta_k^{-2}$, then $\beta_k = 1 + o(\frac{1}{\sqrt{k}})$ as $k\to\infty$.

Write

$$U_k = (u_{ij}^{(k)}) = \sqrt{N-k}\left(\frac{1}{N-k}W_1 - I_p\right),$$

$$V_k = (v_{ij}^{(k)}) = \sqrt{k}\left(\frac{1}{k-1}W_2 - \Sigma_2\right) = \sqrt{k}\left(\frac{1}{k-1}W_2 - \operatorname{diag}[\lambda_1^{(k)},\dots,\lambda_p^{(k)}]\right)$$

Then (3.4) becomes

$$\det\left(\frac{1}{\sqrt{k}}V_k - \frac{\theta\beta_k}{\sqrt{k}}\delta V_k - D_k\right) = 0, \tag{3.5}$$

where

$$D_k = \begin{pmatrix} (\phi-\xi_1^{(k)})I_{\mu_1} & & \\ & \ddots & \\ & & (\phi-\xi_H^{(k)})I_{\mu_H} \end{pmatrix}.$$

By the central limit theorem and Lemma 2.1, we can assume that

$$\begin{aligned} U_k &\to U \triangleq (u_{ij}), \quad \text{a.s.} \\ V_k &\to V \triangleq (v_{ij}), \quad \text{a.s.} \end{aligned} \tag{3.6}$$

where u_{ij} and v_{ij} are all normal variables for all i,j, and, U and V are independent. Besides, if we split the matrices U_k, V_k, U and V into the following blocks

$$U_k = (U_{hg}^{(k)}), \quad V_k = (V_{hg}^{(k)}), \quad U = (U_{hg}), \quad V = (V_{hg}),$$

$$h,g = 1,2,\dots,H,$$

with $U_{hg}^{(k)}$, $V_{hg}^{(k)}$, U_{hg}, V_{hg} being of order $\mu_h \times \mu_g$, then U_{hh}, V_{hh}, $h=1,2,\ldots,H$, are independent and, $\frac{1}{\sqrt{2}} U_{hh}$ and $(2m^2\tilde{\psi}_h^2 + 4(\theta^{-2}+1)\tilde{\psi}_h + 2)^{-1/2} V_{hh}$, $h=1,2,\ldots,H$, are all central Gaussian matrices.

Let $\zeta_i^{(k)} = \sqrt{k}(\delta_i^{(k)} - \lambda_i^{(k)})$, $i=1,2,\ldots,p$. Using the argument used in the section 6, we can prove that the joint distribution of $\zeta_1^{(k)},\ldots,\zeta_p^{(k)}$ tends to that of $\zeta_1,\ldots,\zeta_p$ as $k\to\infty$, where $\zeta_{a_{h-1}+1} \geq \zeta_{a_{h-1}+2} \geq \cdots \geq \zeta_{a_h}$ are the roots of

$$\det(V_{hh}-\theta\xi_h U_{hh}-\zeta I_{\mu_h}) = 0, \quad h=1,2,\ldots,H, \tag{3.7}$$

where

$$\xi_h = 1 + (1+\theta^{-2})\tilde{\psi}_h, \quad h=1,2,\ldots,H.$$

Now $[2m^2\tilde{\psi}_h^2 + 4(\theta^{-2}+1)\tilde{\psi}_h + 2 + 2\xi_h]^{-1/2}(V_{hh} - \theta\xi_h U_{hh})$, $h=1,2,\ldots,H$, are independent central Gaussian matrices. So we have the following

THEOREM 3.1. Suppose that $\underset{\sim}{\alpha}_i$ and $\underset{\sim}{\varepsilon}_{ij}$ in (1.1) are both $p\times 1$ normal vectors for each (i,j), and (3.1)-(3.3) are satisfied. Put

$$\eta_i^{(k)} = \sqrt{k}(\delta_i^{(k)} - \lambda_i^{(k)})/\sqrt{2m^2\tilde{\psi}_h^2+4(\theta^{-2}+1)\tilde{\psi}_h+2+2\xi_h},$$

$$i=a_{h-1}+1,\ldots,a_h, \quad h=1,2,\ldots,H,$$

where $\xi_h = 1 + (1+\theta^{-2})\tilde{\psi}_h$. Then, the joint distribution of $\eta_1^{(k)},\ldots,\eta_p^{(k)}$ tends to a limiting distribution with density

$$\prod_{h=1}^{H} D(\eta_{a_{h-1}+1},\ldots,\eta_{a_h}),$$

where $D(x_1,\ldots,x_\mu)$ has given in Theorem 2.3.

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In this paper, the authors derived asymptotic joint distributions of the eigenvalues of some random matrices which arise under components of covariance model.

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